Galerkin model for Turing patterns on a sphere

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(Received 28 October 2004; published 9 September 2005)

We carry out a Lorenz-like truncation of reaction diffusion systems to investigate pattern formation on a spherical surface. Stability analysis suggests spots to be favored more than stripes in general. Numerical calculations on the truncated model are in agreement with the theoretical predictions.

DOI: 10.1103/PhysRevE.72.036208 PACS number(s): 47.54.+r, 05.70.Ln, 89.75.Kd

I. INTRODUCTION

The study of pattern formation in reaction diffusion systems took off with the historical paper of Turing in 1952 in which he showed that two interacting chemical species with different diffusivities can generate a stable pattern [1]. Since then a number of models have been proposed and applied to different situations [2]. One important motivation for the study of patterns is to simulate and ultimately get to the root of patterns seen in nature. Important biological mechanisms such as morphogenesis are believed to originate from spatial variations of chemicals. But a critical aspect that has been largely neglected is the shape and curvature of the domain and its role in pattern formation. Nature seldom provides flat surfaces that have been widely assumed for calculations till now. The little literature available on the subject mostly focus on numerical integration of particular models on curved domains. Varea *et al.* used a generic reaction diffusion model to numerically investigate patterns on a spherical surface [3,10].

However, in order to understand the general nature of pattern formation on curved domains, it is necessary to employ analytical tools (see [4]). It is the aim of this paper to develop the technique of Galerkin truncation, which has been used with considerable success in the field of hydrodynamic instabilities [5], to study the formation of patterns on a spherical surface. The choice of a spherical surface is not incidental since there are many examples of pattern formation on bulbous surfaces in nature [3]. The structure of viruses and the growth of tumors are some of the major biological applications. Patterns on a sphere may be widely classified as spots and bands (stripes).

The truncated model may be exploited to study the competition between different kinds of patterns and find the parameter ranges in which the system shows interesting behavior. The organization of the paper is as follows: in Sec. II A, we set up a Galerkin model for a popular activator inhibitor model proposed by Gierer and Meinhardt [6]. We then study the competition between spots and stripes by a linear stability analysis. In Sec. II B, we apply the same technique to another model. In Sec. II C we present some numerical results obtained with the truncated model followed by a discussion.

II. GALERKIN PROJECTION

The method of Galerkin projection is used to reduce partial differential equations to a finite set of ordinary differential equations. This requires the expansion of the relevant variables, in this case the concentrations of the species, in a complete set. In general, there would be an infinite number of terms in the expansion. So the trick involves selecting only those modes that are necessary to describe the phenomenon we are investigating, i.e., the series is truncated. The procedure is outlined in the following subsections.

A. GM model on a sphere

We shall consider the well known Gierer Meinhardt model and explore the formation of patterns on a spherical surface (see Fig. 1). A simplified version of the model is given by

$$\frac{\partial A}{\partial t} = D\Delta A + \frac{A^2}{B} - A + \sigma, \tag{1a}$$

$$\frac{\partial B}{\partial t} = \Delta B + \mu(A^2 - B), \tag{1b}$$

where A and B represent the concentrations of two species, D is the ratio of the diffusivities, σ is the basic production rate of A, and μ is the removal rate of B. Δ stands for the Laplace Beltrami operator for a sphere. The general expression for the Laplace Beltrami operator is

$$g(\alpha)^{-1/2}\frac{\partial}{\partial\alpha^i}g(\alpha)^{1/2}g^{ij}(\alpha)\frac{\partial}{\partial\alpha^j}.$$

For a sphere of radius R this is the usual

$$\frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{R^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

The trivial fixed point representing the homogeneous steady state of the system is given by $A_0=(1+\sigma)$ and $B_0=A_0^2$. We carry out a linear stability analysis about this fixed point by taking perturbations of the form

$$\delta A = Y_{l,m}(\theta,\phi)\exp(\lambda t)\delta A_0, \tag{2a}$$

$$\delta B = Y_{l,m}(\theta,\phi)\exp(\lambda t)\delta B_0. \tag{2b}$$

Since we are interested in a spherical domain, the obvious choice of basis functions are spherical harmonics. The traditional k^2 on a flat domain is now replaced by $l(l+1)/R^2$ where l is the polar index and R is the radius of the sphere.

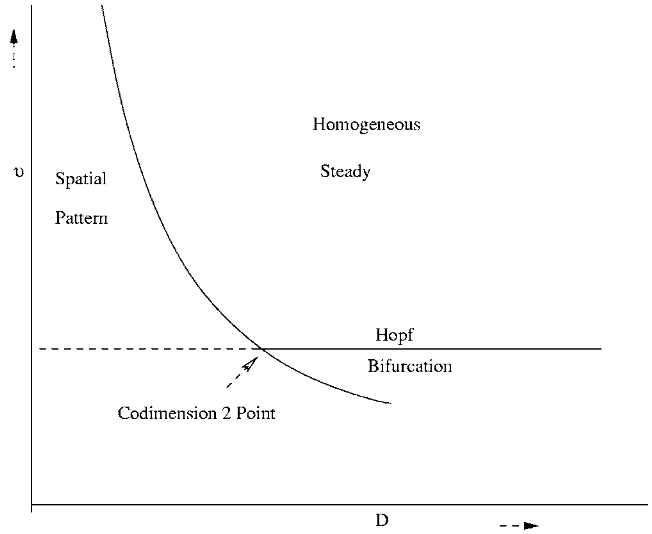

FIG. 1. Stability diagram for Eqs. (1a) and (1b).

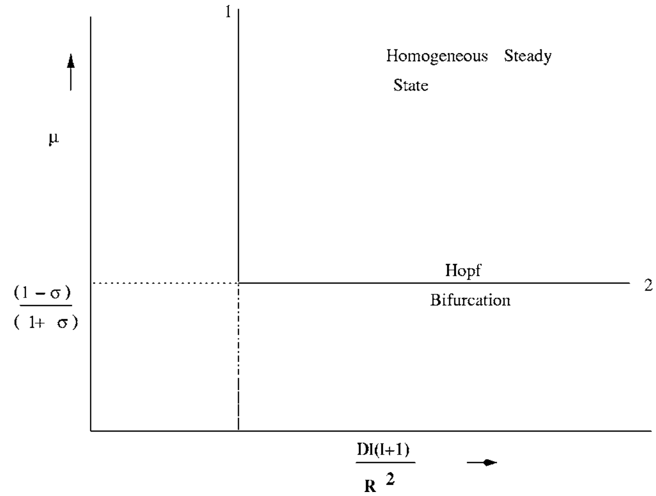

FIG. 2. Linear stability diagram for Eqs. (5a)–(5e). Boundary 2 separates the homogeneous steady state from the homogeneous oscillatory state. On boundary 1, the mode $a_{l,0}$ becomes unstable.

We thus find that the mode having the highest positive growth rate is characterized by

$$\left.\frac{l(l+1)}{R^2}\right|_c = \sqrt{\frac{\mu}{D}}, \quad \text{where } \mu D = \left[\sqrt{\frac{2}{1+\sigma}} - 1\right]^2. \quad (3)$$

The critical value of l is determined by the actual problem, i.e., by the value of R. As the radius increases, so does the polar index of the critical mode which means that the pattern becomes more complicated.

We now set up the Galerkin model for the system keeping the number of modes to a minimum. The modes have to be judiciously chosen so as to capture the features of the phenomenon we are interested in, in this case, the competition between stripes and spots.

We define real combinations of spherical harmonics as follows:

$$y_{l,0}(\theta,\phi) = Y_{l,0}(\theta,\phi),$$

$$y_{l,|m|}(\theta,\phi) = \frac{1}{\sqrt{2}}[Y_{l,-|m|}(\theta,\phi) + (-1)^m Y_{l,|m|}(\theta,\phi)],$$

$$y_{l,-|m|}(\theta,\phi) = \frac{i}{\sqrt{2}}[Y_{l,-|m|}(\theta,\phi) - (-1)^m Y_{l,|m|}(\theta,\phi)].$$

Expanding

$$A = \sqrt{4\pi}[a_{0,0}y_{0,0}(\theta,\phi) + a_{l,0}y_{l,0}(\theta,\phi) + a_{l_1,m}y_{l_1,m}(\theta,\phi) + a_{l_1,-m}y_{l_1,-m}(\theta,\phi)], \quad (4a)$$

$$B = \sqrt{4\pi}[b_{0,0}y_{0,0}(\theta,\phi)], \quad (4b)$$

where for a given radius R, l is the polar index if the mode with the highest growth rate given by Eq. (3). We have chosen a five mode truncation which is the simplest situation. Since B is the fast diffusing species, we assume that the higher modes would decay rapidly. Ideally for a given l, we should keep all the $2l+1$ modes. But the expansions (4a) and (4b) is sufficient to set up a Galerkin model to study the basic aspects of pattern formation. Substituting in Eqs. (1a) and (1b) and using the orthonormality conditions of spherical harmonics, we obtain a set of nonlinear ordinary differential equations:

$$\dot{a}_{0,0} = \sigma - a_{0,0} + \frac{1}{b_{0,0}}(a_{0,0}^2 + a_{l,0}^2 + a_{l_1,m}^2 + a_{l_1,-m}^2), \quad (5a)$$

$$\dot{b}_{0,0} = \mu(-b_{0,0} + a_{0,0}^2 + a_{l,0}^2 + a_{l_1,m}^2 + a_{l_1,-m}^2), \quad (5b)$$

$$\dot{a}_{l,0} = -\left(\frac{Dl(l+1)}{R^2} + 1\right)a_{l,0} + \frac{1}{b_{0,0}y_{0,0}}\left(a_{l,0}^2 \int y_{l,0}^3 d\Omega + \frac{2a_{0,0}a_{l,0}}{\sqrt{4\pi}} + a_{l_1,m}^2 \int y_{l,0}y_{l_1,m}^2 d\Omega + a_{l_1,-m}^2 \int y_{l,0}y_{l_1,-m}^2 d\Omega\right), \quad (5c)$$

$$\dot{a}_{l_1,m} = -\left(\frac{Dl_1(l_1+1)}{R^2} + 1\right)a_{l_1,m} + \frac{1}{b_{0,0}y_{0,0}}\left(\frac{2a_{0,0}a_{l_1,m}}{\sqrt{4\pi}} + 2a_{l,0}a_{l_1,m}\int y_{l,0}y_{l_1,m}^2 d\Omega\right), \quad (5d)$$

$$\dot{a}_{l_1,-m} = -\left(\frac{Dl_1(l_1+1)}{R^2} + 1\right)a_{l_1,-m} + \frac{1}{b_{0,0}y_{0,0}}\left(\frac{2a_{0,0}a_{l_1,-m}}{\sqrt{4\pi}} + 2a_{l,0}a_{l_1,-m}\int y_{l,0}y_{l_1,-m}^2 d\Omega\right). \quad (5e)$$

Now that we have a set of ordinary differential equations (ODE's), we can find out the fixed points of the system (see Fig. 2). The trivial fixed point describing the homogeneous steady state is given by $a_{0,0}=(1+\sigma)$ and $b_{0,0}=(1+\sigma)^2$. A linear stability analysis shows that for $Dl(l+1)/R^2 \leqslant (1-\sigma)/(1+\sigma)$, the trivial fixed point becomes unstable against perturbations $\delta a_{l,0}$. If $l_1 = l$, then the instability in the three modes $a_{l,\pm m}$ and $a_{l,0}$ sets in simultaneously.

We can study the competition between stripes and spots by checking the linear stability of the striped state ($a_{l_1,\pm m} = 0$) against perturbations by $\delta a_{l_1,\pm m}$. The striped fixed point is given by

$$a_{l_1,\pm m} = 0, \tag{6a}$$

$$a_{0,0} - (1+\sigma) = 0, \tag{6b}$$

$$b_{0,0} - (a_{0,0}^2 + a_{l,0}^2) = 0, \tag{6c}$$

$$-\left(\frac{Dl(l+1)}{R^2} + 1\right) + \frac{a_{l,0}}{b_{l,0} y_{0,0}} \int y_{l,0}^3 d\Omega + \frac{2a_{0,0}}{b_{0,0}} = 0. \tag{6d}$$

The linearized equations in the subspace of $a_{l_1,\pm m}$ are given by

$$\delta \dot{a}_{l_1,m} = \left[-\left(\frac{Dl_1(l_1+1)}{R^2} + 1\right) + \frac{2a_{l,0}}{b_{l,0} y_{0,0}} \int y_{l,0} y_{l_1,m}^2 d\Omega + \frac{2a_{0,0}}{b_{0,0}} \right] \delta a_{l_1,m}, \tag{7a}$$

$$\delta \dot{a}_{l_1,-m} = \left[-\left(\frac{Dl_1(l_1+1)}{R^2} + 1\right) + \frac{2a_{l,0}}{b_{l,0} y_{0,0}} \int y_{l,0} y_{l_1,-m}^2 d\Omega + \frac{2a_{0,0}}{b_{0,0}} \right] \delta a_{l_1,-m}. \tag{7b}$$

There are two cases that we can tackle separately.

When l is odd, the condition for the perturbations $\delta a_{l_1,\pm m}$ to grow is

$$-\left(\frac{Dl_1(l_1+1)}{R^2} + 1\right) + \frac{2a_{0,0}}{b_{0,0}} \geqslant 0.$$

Using Eq. (6d),

$$-\left(\frac{Dl_1(l_1+1)}{R^2} + 1\right) + \left(\frac{Dl(l+1)}{R^2} + 1\right) \geqslant 0,$$

or

$$l - l_1 \geqslant 0.$$

The modes $a_{l_1,\pm m}$ are found to have marginal stability (i.e., have zero growth rates) if $l_1 = l$ and when $l_1 < l$, the modes have a positive growth rate.

When l is even, the condition for onset of instability is given by

$$-\left(\frac{Dl_1(l_1+1)}{R^2} + 1\right) + \frac{2a_{0,0}}{b_{0,0}} + \frac{2a_{l,0}}{b_{l,0} y_{0,0}} \int y_{l,0} y_{l_1,\pm m}^2 d\Omega \geqslant 0.$$

Now, using the relation

$$\int y_{l,0}^3 d\Omega = 2 \int y_{l,0} y_{l,\pm 1}^2 d\Omega$$

the modes $a_{l,\pm 1}$ are marginally unstable. Linear stability analysis cannot tell us if the striped state is stable against these perturbations or not.

Hence within the framework of the Galerkin model, a linear stability analysis suggests that stripes are unstable with respect to spots.

B. A generic reaction diffusion model

In order to establish the general applicability of the truncation method, we shall test it on a commonly used model introduced by Barrio *et al.* [8,7,9]. A two component reaction diffusion system has the standard from

$$\frac{\partial \mathbf{U}}{\partial t} = \mathbf{D}\Delta \mathbf{U} + \mathbf{F}(\mathbf{U}), \tag{8}$$

where the diagonal matrix $\mathbf{D}$ gives the diffusion coefficients and $\mathbf{F} = (F_1, F_2)$ describes the (nonlinear) interaction of the two species. Expanding the nonlinear functions around a stationary uniform solution, keeping terms up to cubic order

$$\frac{\partial u}{\partial t} = \delta D \Delta u + \alpha u(1 - r_1 v^2) + v(1 - r_2 u), \tag{9a}$$

$$\frac{\partial v}{\partial t} = \delta \Delta v + \beta v\left(1 + \frac{\alpha r_1}{\beta} uv\right) + u(\gamma + r_2 v), \tag{9b}$$

where u and v are the deviations of the concentration of the two species from the uniform steady state values. Hence the trivial fixed point is given by the point $(u=0, v=0)$. D is the ratio between the diffusion coefficients of the two species. α, β, and γ are the parameters of the system. We set up the Galerkin model for this system, on a sphere, as follows. Let

$$u = A_{0,0} y_{0,0}(\theta,\phi) + A_{l,0} y_{l,0}(\theta,\phi) + A_{l,m} y_{l,m}(\theta,\phi) + A_{l,-m} y_{l,-m}(\theta,\phi),$$

$$v = B_{0,0} y_{0,0}(\theta,\phi) + B_{l,0} y_{l,0}(\theta,\phi) + B_{l,m} y_{l,m}(\theta,\phi) + B_{l,-m} y_{l,-m}(\theta,\phi).$$

Keeping only the quadratic coupling, i.e., setting $r_1 = 0$, we obtain

$$\dot{A}_{0,0} = \alpha A_{0,0} + B_{0,0} - r_2(B_{0,0}A_{0,0} + B_{l,0}A_{l,0} + B_{l,m}A_{l,m} + B_{l,-m}A_{l,-m}), \tag{10a}$$

$$\dot{B}_{0,0} = \gamma A_{0,0} + \beta B_{0,0} + r_2(B_{0,0}A_{0,0} + B_{l,0}A_{l,0} + B_{l,m}A_{l,m} + B_{l,-m}A_{l,-m}), \tag{10b}$$

$$\dot{A}_{l,0} = \alpha A_{l,0} + B_{l,0} - \frac{\delta D l(l+1)}{R^2} A_{l,0} - r_2 \left(\frac{B_{0,0} A_{l,0}}{\sqrt{4\pi}} + \frac{B_{l,0} A_{0,0}}{\sqrt{4\pi}} \right) - r_2 \left(B_{l,0} A_{l,0} \int y_{l,0}^3 d\Omega + B_{l,m} A_{l,m} \int y_{l,m}^2 y_{l,0} d\Omega + B_{l,-m} A_{l,-m} \int y_{l,-m}^2 y_{l,0} d\Omega \right), \quad (10c)$$

$$\dot{B}_{l,0} = \gamma A_{l,0} + \beta B_{l,0} - \frac{\delta l(l+1)}{R^2} B_{l,0} + r_2 \left(\frac{B_{0,0} A_{l,0}}{\sqrt{4\pi}} + \frac{B_{l,0} A_{0,0}}{\sqrt{4\pi}} \right) + r_2 \left(B_{l,0} A_{l,0} \int y_{l,0}^3 d\Omega + B_{l,m} A_{l,m} \int y_{l,m}^2 y_{l,0} d\Omega + B_{l,-m} A_{l,-m} \int y_{l,-m}^2 y_{l,0} d\Omega \right), \quad (10d)$$

$$\dot{A}_{l,\pm m} = \left(-\frac{\delta D l(l+1)}{R^2} + \alpha \right) A_{l,\pm m} + B_{l,\pm m} - r_2 \left(\frac{B_{0,0} A_{l,\pm m} + A_{0,0} B_{l,\pm m}}{\sqrt{4\pi}} + (B_{l,0} A_{l,\pm m} + A_{l,0} B_{l,\pm m}) \int y_{l,\pm m}^2 y_{l,0} d\Omega \right), \quad (10e)$$

$$\dot{B}_{l,\pm m} = \left(-\frac{\delta l(l+1)}{R^2} + \beta \right) B_{l,\pm m} + \gamma A_{l,\pm m} + r_2 \left(\frac{B_{0,0} A_{l,\pm m} + A_{0,0} B_{l,\pm m}}{\sqrt{4\pi}} + (B_{l,0} A_{l,\pm m} + A_{l,0} B_{l,\pm m}) \int y_{l,\pm m}^2 y_{l,0} d\Omega \right). \quad (10f)$$

A linear stability analysis shows the striped state ($A_{l,\pm m} = B_{l,\pm m} = 0$) to be unstable to spots. A similar analysis may be carried out keeping only the cubic coupling and setting $r_2 = 0$. But due to practical difficulties in keeping l and m general, it is necessary to use the actual values of l and m. For $l=1$ and $m=1$, the striped state was found to be unstable to spots. Thus stripes appear to be generically unstable.

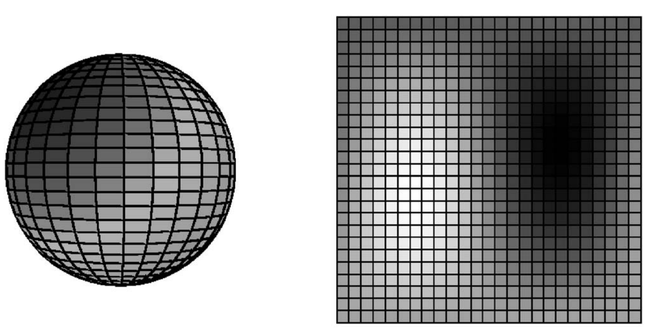

FIG. 3. Pattern of A drawn with a linear grey scale obtained for $l=1$, $Dl(l+1)/R^2=0.2$, $\sigma=0.5$, and $\mu=1.0$.

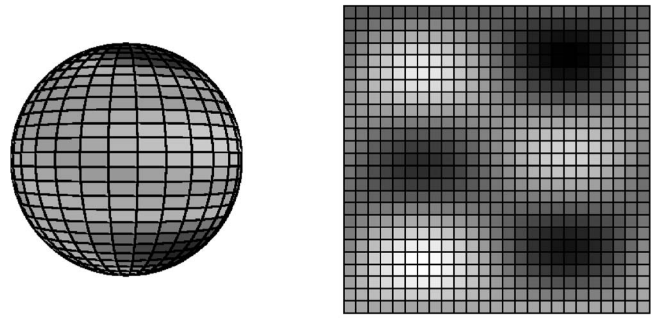

FIG. 4. Pattern of A with $l=3$, same parameters.

C. Numerical results

We have carried out numerical calculations for the truncated system obtained for the Gierer Meinhardt model. For odd values of l, stripes were found to be unstable. As predicted by the linear stability analysis, for $Dl(l+1)/R^2$ less than the threshold value $(1-\sigma)/(1+\sigma)$, the concentration profile of the species A showed interesting patterns, some of which are sketched in Fig 3.

However, there is a basic difference between even and odd values of l. For odd values of l, some of the quadratic coupling terms vanish. But for even values of l, they serve to break the degeneracy among the equations for the $2l+1$ modes. For $l=2$ and $m=1$, linear stability analysis failed to predict if stripes are unstable to spots or not. A numerical calculation of Eqs. (5a)–(5e) suggests that for $l=2$ and $m=1$, stripes are *stable*. But to get a clearer picture it is necessary to keep all the modes from $m=-2$ to $m=2$ in the expansion of A (see Fig. 4). Hence we expand

$$A = \sqrt{4\pi}(a_{0,0} y_{0,0}(\theta,\phi) + a_{2,-2} y_{2,-2}(\theta,\phi) + a_{2,-1} y_{2,-1}(\theta,\phi)) + a_{2,0} y_{1,0}(a_{2,0} y_{l,0}(\theta,\phi) + a_{2,1} y_{2,1}(\theta,\phi) + a_{2,2} y_{2,2}(\theta,\phi)), \quad (11a)$$

$$B = \sqrt{4\pi}[b_{0,0} y_{0,0}(\theta,\phi)]. \quad (11b)$$

By solving the complete set of equations obtained by substituting Eqs. (11a) and (11b) in Eq. (1b), we obtain a spotted pattern (Fig. 5).

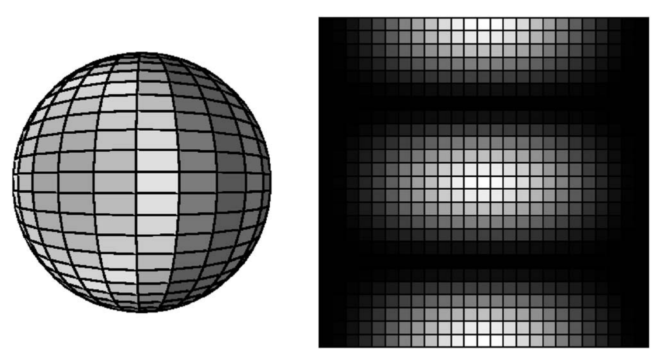

FIG. 5. Pattern of A with $l=2$, $Dl(l+1)/R^2=0.3$, $\sigma=0.5$, and $\mu=1.0$.

III. DISCUSSION

We have been able to set up a truncated model for two different reaction diffusion models. This vastly simplifies the job of numerical calculations and makes it possible to get an idea of the patterns analytically. We studied the competition between stripes and spots. The term "stripes" has to be used with caution as one cannot obtain perfect stripes on a sphere because of its topology. There would have to be defects [3]. For instance, a pattern of rings around the surface would end in two spots at the poles. Hence patterns on a sphere cannot be directly compared to those on a plane.

Linear stability analysis suggests that spots are favored more than stripes in general. This was confirmed by numerical calculations on the truncated version of the Gierer Meinhardt model. An obvious problem with this technique is that it is possible to miss the relevant modes since it requires a prior idea of the pattern. But if guided by experimental results, it can be an effective tool. The Galerkin model can be used to investigate patterns far from the major phase boundaries. The model can be extended to probe other aspects of pattern formation such as Hopf bifurcation and the Hopf-Turing mixed region.

Pattern formation on curved surfaces needs to be probed in greater detail. The Galerkin truncation technique is a useful tool that can reveal interesting phenomena analytically which are otherwise not obvious.

ACKNOWLEDGMENT

The author wishes to thank Professor J. K. Bhattacharjee for his support.

APPENDIX

In this appendix, we will deal with the calculations related to Eqs. (9) and (10). Keeping only the quadratic coupling, the striped state is given by

$$\alpha A_{0,0} + B_{0,0} - r_2(B_{0,0}A_{0,0} + B_{l,0}A_{l,0}) = 0, \tag{A1a}$$

$$\beta B_{0,0} + \gamma A_{0,0} + r_2(B_{0,0}A_{0,0} + B_{l,0}A_{l,0}) = 0, \tag{A1b}$$

$$\left[\alpha - \frac{\delta D l(l+1)}{R^2}\right]A_{l,0} + B_{l,0} - r_2\left[\frac{B_{0,0}A_{l,0} + B_{l,0}A_{0,0}}{\sqrt{4\pi}} + B_{l,0}A_{l,0}\int y_{l,0}^3 d\Omega\right] = 0, \tag{A1c}$$

$$\left[\beta - \frac{\delta l(l+1)}{R^2}\right]B_{l,0} + \gamma A_{l,0} + r_2\left[\frac{B_{0,0}A_{l,0} + B_{l,0}A_{0,0}}{\sqrt{4\pi}} + B_{l,0}A_{l,0}\int y_{l,0}^3 d\Omega\right] = 0, \tag{A1d}$$

$$A_{l,\pm m} = 0, \tag{A1e}$$

$$B_{l,\pm m} = 0. \tag{A1f}$$

Taking perturbations $\delta A_{l,m}$ and $\delta B_{l,m}$, the linearized equations are

$$\begin{pmatrix} \delta\dot{A}_{l,m} \\ \delta\dot{B}_{l,m} \end{pmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{pmatrix} \delta A_{l,m} \\ \delta B_{l,m} \end{pmatrix}, \tag{A2}$$

where

$$c_{11} = \alpha - \frac{\delta D l(l+1)}{R^2} - r_2\left(\frac{B_{0,0}}{\sqrt{4\pi}} + B_{l,0}\int y_{l,m}^2 y_{l,0} d\Omega\right),$$

$$c_{12} = 1 - r_2\left(\frac{A_{0,0}}{\sqrt{4\pi}} + A_{l,0}\int y_{l,m}^2 y_{l,0} d\Omega\right),$$

$$c_{21} = \gamma + r_2\left(\frac{B_{0,0}}{\sqrt{4\pi}} + B_{l,0}\int y_{l,m}^2 y_{l,0} d\Omega\right),$$

$$c_{22} = \beta - \frac{\delta l(l+1)}{R^2} + r_2\left(\frac{A_{0,0}}{\sqrt{4\pi}} + A_{l,0}\int y_{l,m}^2 y_{l,0} d\Omega\right).$$

Assuming solutions of the form

$$\begin{pmatrix} \delta A_{l,m}(t) \\ \delta B_{l,m}(t) \end{pmatrix} = \exp(\lambda t)\begin{pmatrix} \delta A_{l,m}(0) \\ \delta B_{l,m}(0) \end{pmatrix},$$

the perturbations would grow if $\mathrm{Re}\lambda > 0$, and the condition for the onset of stationary instability in the subspace of $A_{l,m}$ and $B_{l,m}$ is $\lambda = 0$. This gives $c_{11}c_{22} - c_{12}c_{21} = 0$, or

$$\left[\alpha - \frac{\delta D l(l+1)}{R^2} - r_2\left(\frac{B_{0,0}}{\sqrt{4\pi}} + B_{l,0}\int y_{l,m}^2 y_{l,0} d\Omega\right)\right] \times \left[\beta - \frac{\delta l(l+1)}{R^2} + r_2\left(\frac{A_{0,0}}{\sqrt{4\pi}} + A_{l,0}\int y_{l,m}^2 y_{l,0} d\Omega\right)\right] - \left[1 - r_2\left(\frac{A_{0,0}}{\sqrt{4\pi}} + A_{l,0}\int y_{l,m}^2 y_{l,0} d\Omega\right)\right] \times \left[\gamma + r_2\left(\frac{B_{0,0}}{\sqrt{4\pi}} + B_{l,0}\int y_{l,m}^2 y_{l,0} d\Omega\right)\right] = 0. \tag{A3}$$

For $m = 1$,

$$\int y_{l,0}^3 d\Omega = 2\int y_{l,0} y_{l,1}^2 d\Omega \tag{A4}$$

if l is even. If l is odd, these integrals vanish. From Eqs. (A1c) and (A4)

$$\left[\alpha-\frac{\delta Dl(l+1)}{R^2}-r_2\left(\frac{B_{0,0}}{\sqrt{4\pi}}+B_{l,0}\int y_{l,1}^2 y_{l,0}d\Omega\right)\right]A_{l,0}+\left[1-r_2\left(\frac{A_{0,0}}{\sqrt{4\pi}}+A_{l,0}\int y_{l,1}^2 y_{l,0}d\Omega\right)\right]B_{l,0}=0. \quad \text{(A5)}$$

Again from Eq. (A1d),

$$\left[\gamma+r_2\left(\frac{B_{0,0}}{\sqrt{4\pi}}+B_{l,0}\int y_{l,0}y_{l,1}^2 d\Omega\right)\right]A_{l,0}\times\left[\beta-\frac{\delta l(l+1)}{R^2}+r_2\left(\frac{A_{0,0}}{\sqrt{4\pi}}+A_{l,0}\int y_{l,0}y_{l,1}^2 d\Omega\right)\right]B_{l,0}=0. \quad \text{(A6)}$$

From Eqs. (A5) and (A6)

$$-\frac{A_{l,0}}{B_{l,0}}=\frac{\left[1-r_2\left(\frac{A_{0,0}}{\sqrt{4\pi}}+A_{l,0}\int y_{l,1}^2 y_{l,0}d\Omega\right)\right]}{\left[\alpha-\frac{\delta Dl(l+1)}{R^2}-r_2\left(\frac{B_{0,0}}{\sqrt{4\pi}}+B_{l,0}\int y_{l,1}^2 y_{l,0}d\Omega\right)\right]}=\frac{\left[\beta-\frac{\delta l(l+1)}{R^2}+r_2\left(\frac{A_{0,0}}{\sqrt{4\pi}}+A_{l,0}\int y_{l,0}y_{l,1}^2 d\Omega\right)\right]}{\left[\gamma+r_2\left(\frac{B_{0,0}}{\sqrt{4\pi}}+B_{l,0}\int y_{l,0}y_{l,1}^2 d\Omega\right)\right]}. \quad \text{(A7)}$$

Thus we see that Eq. (A3), which is the condition for marginal instability, i.e., $\lambda=0$, is satisfied. A similar calculation can be carried out for cubic coupling, with $r_2=0$. Here it is necessary to use specific values of l and m.

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